JOURNAL OF COMPUTATIONAL PHYSICS 6, 483–509 (1970)

Finite Difference Methods for the Fokker–Planck Equation*

JAMES C. WHITNEY

Columbia University, New York, New York 10027

Received December 11, 1969

Two methods are developed for numerical solution of the time-dependent Fokker–Planck equation. First, a finite difference scheme is constructed which has the property of simultaneously conserving total system mass, energy, and momentum throughout the interior of velocity space, thereby reducing nonconservation errors to boundary contributions only. Second, the Fokker–Planck collision operator is expanded in Legendre polynomials and reduced to an equivalent system of one-dimensional problems. The expansion is carried out to all orders and is mathematically exact; no terms are neglected or linearized. The resulting system of one-dimensional equations is given explicitly. Both these formulations are given for the case of two-dimensional azimuthally symmetric distribution functions. As a test of the methods, the thermal equipartition of an isotropic deuterium–electron plasma is computed numerically. Over a range of initial conditions from $T_e/T_D = 0.01$ to $T_e/T_D = 100$ the electron and ion temperatures obtained from the Fokker–Planck equation agree to within 6 % with those obtained from the Maxwellian model given by Spitzer.

1. INTRODUCTION

Numerical solutions of the time-dependent Fokker–Planck equation have been used for many years to study collisional processes in plasmas. Among the many problems encountered in solution of this equation, two are of particular importance. First, the previous numerical treatments [1–7] have used difference schemes which violate the conservation laws of mass, energy, and momentum, even though these are intrinsic properties of the equation itself. Nonconservative schemes can introduce significant error unless the grid spacing is chosen very fine. Second, the Fokker–Planck collision operator is very complex in more than one dimension. The multi-dimensional operator can, in principle, be reduced to a coupled set of one-dimensional problems by expanding the angular dependence in spherical harmonics. Although the first two terms of this expansion, in a somewhat simplified form, have been used by several authors [8–10], the correct and explicit form of the general term has not previously been obtained.

* Part of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, in the Faculty of Pure Science, Columbia University.

581/6/3-10

In Section 2 of this paper we develop a finite difference scheme for the two-dimensional axially symmetric Fokker–Planck equation which has the property of conserving total system mass, energy, and momentum throughout the interior of velocity space independent of the mesh. Any conservation errors are then due only to points on the boundaries. If desired, the boundary errors for any one of these moments (but not all three together) can be eliminated by suitable choice of boundary conditions. Even so, we can reasonably expect that the boundary errors will be extremely small when compared to the error which accumulates throughout the whole velocity space for a nonconservative scheme. Thus the conservative scheme offers a significant increase in accuracy and pays dividends for those problems in which it has been necessary to restrict the grid spacing to keep conservation errors small.

In Section 3 we reduce the two-dimensional Fokker–Planck equation to a system of one-dimensional equations by expanding the collision operator to all orders in Legendre polynomials. No terms are neglected or linearized. The final system of equations is expressed in a compact form which is useful for numerical calculations. While the one-dimensional schemes do not have the conservation properties of the more exact two-dimensional method, they greatly reduce the amount of computation and are well suited for problems in which the distribution functions are close to isotropy.

In Section 4 we apply these numerical methods to compute the long-time equipartition of a deuterium–electron plasma in which $T_e \neq T_D$. We find that over a range of initial conditions from $T_e/T_D = 0.01$ to $T_e/T_D = 100$ the simple relaxation model of Spitzer [11] (which assumes Maxwellian distribution functions) gives relaxation times in remarkably close agreement with the Fokker–Planck results.

2. Conservative Difference Scheme

The Fokker–Planck equation for a spatially uniform plasma has the form

$$\frac{\partial f_\alpha}{\partial t} = C_\alpha(f), \tag{1}$$

where the subscript α denotes particle species. In Cartesian coordinates the collision operator C_α is [12]

$$C_\alpha = -\frac{\partial}{\partial v_i}\left(f_\alpha \frac{\partial H_\alpha}{\partial v_i}\right) + \frac{1}{2}\frac{\partial^2}{\partial v_i\, \partial v_j}\left(f_\alpha \frac{\partial^2 G_\alpha}{\partial v_i\, \partial v_j}\right), \tag{2}$$

where

$$H_\alpha(\mathbf{v}) = \Gamma_\alpha \sum_\beta \left(1 + \frac{m_\alpha}{m_\beta}\right) h_\beta(\mathbf{v}), \tag{3a}$$

$$G_\alpha(\mathbf{v}) = \Gamma_\alpha \sum_\beta g_\beta(\mathbf{v}), \tag{3b}$$

$$h_\beta(\mathbf{v}) = \int |\mathbf{v} - \mathbf{v}'|^{-1} f_\beta(\mathbf{v}')\, d\mathbf{v}', \tag{3c}$$

$$g_\beta(\mathbf{v}) = \int |\mathbf{v} - \mathbf{v}'|\, f_\beta(\mathbf{v}')\, d\mathbf{v}', \tag{3d}$$

$$\Gamma_\alpha = \frac{4\pi e^4}{m_\alpha^2} \ln \Lambda. \tag{3e}$$

The β summation in Eqs. (3) includes all particle species. The quantity Λ is the ratio of Debye length to the mean impact parameter for 90° scattering [11],

$$\Lambda = \frac{3}{2e^3}\left(\frac{k^3T^3}{\pi n}\right)^{1/2}.$$

The conservation laws for mass, momentum, and energy are expressed by the relations

$$\begin{aligned} \int C_\alpha(f)\, d\mathbf{v} &= 0, \\ \sum_\alpha \int m_\alpha \mathbf{v} C_\alpha(f)\, d\mathbf{v} &= 0, \\ \sum_\alpha \int \tfrac{1}{2} m_\alpha |\mathbf{v}|^2 C_\alpha(f)\, d\mathbf{v} &= 0. \end{aligned} \tag{4}$$

It is not difficult to show that Eq. (1) has these properties. However, when the Fokker–Planck equation is solved numerically the collision operator is replaced by a finite difference approximation $(C_\alpha)_j$ defined on a set of discrete points $\mathbf{v}_j$. Integrals are replaced by finite sums. The conservation laws then take the discrete forms

$$\begin{aligned} \sum_j (C_\alpha)_j\, \Delta\mathbf{v} &= 0, \\ \sum_\alpha \sum_j m_\alpha \mathbf{v}_j (C_\alpha)_j\, \Delta\mathbf{v} &= 0, \\ \sum_\alpha \sum_j \tfrac{1}{2} m_\alpha |\mathbf{v}_j|^2 (C_\alpha)_j\, \Delta\mathbf{v} &= 0. \end{aligned} \tag{5}$$

Unfortunately, the constraints (5) cannot be rigorously satisfied by any finite difference approximation. In general the right sides of Eqs. (5) are nonzero and consist of two sorts of terms: Contributions from the interior points of velocity space and contributions from the boundaries. The boundary terms are unavoidable; they are a consequence of using a finite and discrete velocity space. The boundary terms associated with any one of the equations (5) can be made to vanish by choosing suitable boundary conditions on the difference scheme. Boundary terms from the other two equations will remain. However, these terms are a very small part of the total conservation error. The great bulk of the error comes from points in the interior of the space and arises because the elementary fluxes of mass,

momentum, and energy between different elements of velocity space do not, in general, cancel unless the cell size becomes vanishingly small.

In this section we develop a discrete representation of the collision operator in which the interior errors vanish regardless of the cell size. The conservation laws are properties of the difference scheme in the same way that Eqs. (4) are properties of the exact Fokker–Planck equation. The equations (5) are then satisfied except for the small boundary terms. To construct this scheme we transform the collision operator to spherical coordinates and write it in a particular conservation law form. This conservation form is then descretized to yield a conservative difference scheme.

Conservation Form of the Collision Operator

We first write Eq. (1) in dimensionless form by defining the dimensionless quantities

$$\begin{aligned} \boldsymbol{\nu} &= \mathbf{v}/v_0 , \\ \tau &= t/t_0 , \\ \hat{f}_\alpha &= \frac{{v_0}^3}{c_\alpha} f_\alpha . \end{aligned} \tag{6a}$$

The constants v_0 and c_α are determined from the initial conditions $\hat{f}(\boldsymbol{\nu}, 0)$ by using the definitions of number density and energy,

$$\begin{aligned} c_\alpha &= n_\alpha(0)/I_\alpha(0), \\ {v_0}^2 &= \frac{E_t(0)}{\sum_\alpha \frac{1}{2} m_\alpha n_\alpha(0)[J_\alpha(0)/I_\alpha(0)]} . \end{aligned} \tag{6b}$$

Here $n_\alpha(0)$ and $E_t(0)$ are the initial values of number density and total system energy. The dimensionless quantities I_α and J_α are

$$I_\alpha(\tau) = \int \hat{f}_\alpha(\boldsymbol{\nu}, \tau)\, d\boldsymbol{\nu},$$

$$J_\alpha(\tau) = \int |\, \boldsymbol{\nu} \,|^2 \hat{f}_\alpha(\boldsymbol{\nu}, \tau)\, d\boldsymbol{\nu}.$$

The constant t_0 in Eq. (6) is defined by

$$t_0 = {v_0}^3/\Gamma_e c_e , \tag{7}$$

which is approximately the electron–electron collision time [11]. In terms of these dimensionless variables the Fokker–Planck equation becomes

$$\frac{\partial \hat{f}_\alpha}{\partial \tau} = -\frac{\partial}{\partial \nu_i}\left(\hat{f}_\alpha \frac{\partial \hat{H}_\alpha}{\partial \nu_i}\right) + \frac{1}{2}\frac{\partial^2}{\partial \nu_i\, \partial \nu_j}\left(\hat{f}_\alpha \frac{\partial^2 \hat{G}_\alpha}{\partial \nu_i\, \partial \nu_j}\right). \tag{8}$$

The dimensionless functions $\hat{H}_\alpha$ and $\hat{G}_\alpha$ are given by

$$\begin{aligned} \hat{H}_\alpha(\mathbf{v}) &= \sum_\beta k_{\alpha\beta}\hat{h}_\beta(\mathbf{v}), \\ \hat{G}_\alpha(\mathbf{v}) &= \sum_\beta r_{\alpha\beta}\hat{g}_\beta(\mathbf{v}), \end{aligned} \tag{9a}$$

where

$$\begin{aligned} r_{\alpha\beta} &= (m_e/m_\alpha)^2\,(c_\beta/c_e), \\ k_{\alpha\beta} &= \left(1 + \frac{m_\alpha}{m_\beta}\right) r_{\alpha\beta}\,, \end{aligned} \tag{9b}$$

and

$$\begin{aligned} \hat{h}_\beta(\mathbf{v}) &= \int \hat{f}_\beta(\mathbf{v}')\,|\,\mathbf{v} - \mathbf{v}'\,|^{-1}\,d\mathbf{v}' \\ \hat{g}_\beta(\mathbf{v}) &= \int \hat{f}_\beta(\mathbf{v}')\,|\,\mathbf{v} - \mathbf{v}'\,|\,d\mathbf{v}'\,, \end{aligned} \tag{9c}$$

We shall hereafter omit the caret and regard all quantities as dimensionless.

We next transform Eq. (8) to spherical polar coordinates in velocity space. Since the algebra is straightforward we will not give the details here [13]. The transformed equation is written in terms of the variables $v = |\,\mathbf{v}\,|$ and $\mu = \cos\theta$, where θ is the polar angle in velocity space. We also define the dimensionless functions

$$u_\alpha(v, \mu, \tau) = v^2 f_\alpha(v, \mu, \tau).$$

Then the azimuthally symmetric Fokker–Planck equation can be written in the conservation form [13]

$$\begin{aligned} \frac{\partial}{\partial\tau}\,u_\alpha(v, \mu, \tau) = {} & \frac{\partial^2}{\partial v^2}\,(u_\alpha K_\alpha) + \frac{\partial}{\partial v}\left(\frac{u_\alpha K_\alpha}{v}\right) - \frac{\partial}{\partial v}\left(\frac{u_\alpha P_\alpha}{v}\right) + \frac{\partial}{\partial\mu}\,(u_\alpha D_\alpha) \\ & + \frac{\partial^2}{\partial\mu^2}\,(u_\alpha R_\alpha) + \frac{\partial^2}{\partial v\,\partial\mu}\,(u_\alpha S_\alpha). \end{aligned} \tag{10}$$

The Fokker–Planck coefficients K_α, P_α, D_α, R_α, and S_α are given by

$$\begin{aligned} K_\alpha &= \frac{1}{2}\,\frac{\partial^2 G_\alpha}{\partial v^2}\,, \\ P_\alpha &= v\,\frac{\partial H_\alpha}{\partial v} + \sum_\beta r_{\alpha\beta}h_\beta\,, \end{aligned} \tag{11}$$

$$D_\alpha = \frac{\mu}{\nu^2}\left[\sum_\beta r_{\alpha\beta}h_\beta - \frac{1}{2}\frac{\partial^2 G_\alpha}{\partial \nu^2}\right] - \frac{(1-\mu^2)}{\nu^2}\left[\frac{1}{\nu^2}\frac{\partial G_\alpha}{\partial \mu} - \frac{1}{\nu}\frac{\partial^2 G_\alpha}{\partial \nu\,\partial \mu} + \frac{\partial H_\alpha}{\partial \mu}\right],$$

$$R_\alpha = \frac{(1-\mu^2)}{2\nu^4}\frac{\partial}{\partial \mu}\left[(1-\mu^2)\frac{\partial G_\alpha}{\partial \mu}\right] + \frac{\mu(1-\mu^2)}{2\nu^4}\frac{\partial G_\alpha}{\partial \mu} + \frac{(1-\mu^2)}{2\nu^3}\frac{\partial G_\alpha}{\partial \nu},$$

$$S_\alpha = \frac{(1-\mu^2)}{\nu^2}\left[\frac{\partial^2 G_\alpha}{\partial \nu\,\partial \mu} - \frac{1}{\nu}\frac{\partial G_\alpha}{\partial \mu}\right].$$

Since the right side of Eq. (10) is a pure divergence in velocity space, we can obtain a mass-conserving difference scheme by simply replacing the derivatives with centered finite differences. However, such a scheme will not conserve energy or momentum unless the coefficients (11) are computed in a particular way. Our method is to expand the collision integrals in Legendre polynomials, differentiate the expansions, and substitute the results into Eqs. (11). We can then discretize (10) in such a way that all conservation laws are satisfied. The expansions of the integrals (9c), in the case of azimuthal symmetry, have been given by Rosenbluth, MacDonald, and Judd [12],

$$h_\beta(\nu, \mu) = \sum_{l=0}^{\infty} P_l(\mu)\, h_\beta{}^l(\nu),$$

$$g_\beta(\nu, \mu) = \sum_{l=0}^{\infty} P_l(\mu)\, g_\beta{}^l(\nu), \tag{12}$$

where

$$h_\beta{}^l(\nu) = \frac{4\pi}{(2l+1)}\left[\nu^{-l-1}\int_0^\nu \nu'^l u_\beta{}^l(\nu')\,d\nu' + \nu^l\int_\nu^\infty \nu'^{-l-1}u_\beta{}^l(\nu')\,d\nu'\right],$$

$$g_\beta{}^l(\nu) = \frac{4\pi}{(2l+1)}\left[\frac{1}{(2l+3)}\left(\nu^{-l-1}\int_0^\nu \nu'^{l+2}u_\beta{}^l(\nu')\,d\nu' + \nu^{l+2}\int_\nu^\infty \nu'^{-l-1}u_\beta{}^l(\nu')\,d\nu'\right)\right.$$

$$\left. - \frac{1}{(2l-1)}\left(\nu^{1-l}\int_0^\nu \nu'^l u_\beta{}^l(\nu')\,d\nu' + \nu^l\int_\nu^\infty \nu'^{1-l}u_\beta{}^l(\nu')\,d\nu'\right)\right], \tag{14}$$

$$u_\beta{}^l(\nu') = \frac{2l+1}{2}\int_{-1}^{+1} P_l(\mu')\, u_\beta(\nu', \mu')\,d\mu'. \tag{14}$$

The Fokker–Planck coefficients (11) now follow directly from Eqs. (13) and (14),

$$K_\alpha(\nu, \mu) = \sum_{l=0}^{\infty} P_l(\mu)\sum_\beta r_{\alpha\beta}\frac{1}{2}\frac{\partial^2}{\partial \nu^2}g_\beta{}^l, \tag{15a}$$

$$P_\alpha(\nu, \mu) = \sum_{l=0}^{\infty} P_l(\mu)\sum_\beta \left(k_{\alpha\beta}\nu\frac{\partial}{\partial \nu}h_\beta{}^l + r_{\alpha\beta}h_\beta{}^l\right), \tag{15b}$$

$$S_\alpha(\nu, \mu) = \sum_{l=0}^{\infty} (P_{l-1} - P_{l+1}) \frac{l(l+1)}{(2l+1)} \sum_\beta r_{\alpha\beta} \left(\frac{1}{\nu^2} \frac{\partial}{\partial \nu} g_\beta{}^l - \frac{1}{\nu^3} g_\beta{}^l \right), \tag{15c}$$

$$D_\alpha(\nu, \mu) = \sum_{l=0}^{\infty} \sum_\beta \left\{ \mu P_l(\mu)\, r_{\alpha\beta} \left[\frac{1}{\nu^2} h_\beta{}^l - \frac{1}{2\nu^2} \frac{\partial^2}{\partial \nu^2} g_\beta{}^l \right] + \frac{l(l+1)}{(2l+1)} (P_{l-1} - P_{l+1}) \left[r_{\alpha\beta} \left(\frac{1}{\nu^3} \frac{\partial}{\partial \nu} g_\beta{}^l - \frac{1}{\nu^4} g_\beta{}^l \right) - k_{\alpha\beta} \frac{1}{\nu^2} h_\beta{}^l \right] \right\}, \tag{15d}$$

$$R_\alpha(\nu, \mu) = \sum_{l=0}^{\infty} \sum_\beta r_{\alpha\beta} \left\{ \frac{l(l+1)}{(2l+1)} \left[\frac{(l+1)(l+2)}{(2l+3)} P_{l+2} + \mu P_{l+1} - \mu P_{l-1} - \left(\frac{l(l-1)}{(2l-1)} + \frac{(l+1)(l+2)}{(2l+3)} \right) P_l + \frac{l(l-1)}{(2l-1)} P_{l-2} \right] \frac{1}{2\nu^4} g_\beta{}^l + (1 - \mu^2) P_l \frac{1}{2\nu^3} \frac{\partial}{\partial \nu} g_\beta{}^l \right\}, \tag{15e}$$

where

$$\frac{\partial}{\partial \nu} h_\beta{}^l = \frac{4\pi}{(2l+1)} \left\{ l\nu^{l-1} \int_\nu^\infty \nu'^{-l-1} u_\beta{}^l(\nu')\, d\nu' - (l+1)\, \nu^{-l-2} \int_0^\nu \nu'^{l} u_\beta{}^l(\nu')\, d\nu' \right\},$$

$$\frac{\partial}{\partial \nu} g_\beta{}^l = \frac{4\pi}{(2l+1)} \left\{ \frac{1}{(2l+3)} \left[(l+2)\, \nu^{l+1} \int_\nu^\infty \nu'^{-l-1} u_\beta{}^l(\nu')\, d\nu' - (l+1)\, \nu^{-l-2} \int_0^\nu \nu'^{l+2} u_\beta{}^l(\nu')\, d\nu' \right] - \frac{1}{(2l-1)} \left[l\nu^{l-1} \int_\nu^\infty \nu'^{1-l} u_\beta{}^l(\nu')\, d\nu' - (l-1)\, \nu^{-l} \int_0^\nu \nu'^{l} u_\beta{}^l(\nu')\, d\nu' \right] \right\},$$

$$\frac{\partial^2}{\partial \nu^2} g_\beta{}^l = \frac{4\pi}{(2l+1)} \left\{ \frac{(l+1)(l+2)}{(2l+3)} \left[\nu^{-l-3} \int_0^\nu \nu'^{l+2} u_\beta{}^l(\nu')\, d\nu' + \nu^l \int_\nu^\infty \nu'^{-l-1} u_\beta{}^l(\nu')\, d\nu' \right] - \frac{l(l-1)}{(2l-1)} \left[\nu^{-l-1} \int_0^\nu \nu'^{l} u_\beta{}^l(\nu')\, d\nu' + \nu^{l-2} \int_\nu^\infty \nu'^{1-l} u_\beta{}^l(\nu')\, d\nu' \right] \right\}. \tag{16}$$

Finite Difference Scheme

When the representation of the Fokker–Planck equation given by Eqs. (10), (15), and (16) is descretized the conservation laws are preserved independent of the mesh size. We define the distribution functions at the discrete points

$$\begin{aligned} \mu_i &= i(\Delta\mu), \qquad i = -M, \ldots, 0, \ldots, M; \qquad M(\Delta\mu) = 1, \\ \nu_j &= j(\Delta\nu), \qquad j = 1, 2, \ldots, N+1, \\ \tau^n &= n(\Delta\tau), \qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{17}$$

Figure 1 shows the coordinate grid. The velocity space consists of a set of *interior points*

$$\begin{aligned} i &= -M + 1,\ldots, M - 1, \\ j &= 2,\ldots, N, \end{aligned} \tag{18}$$

and a set of boundary points, indicated by dashed lines in Fig. 1. The difference scheme is a closed system of equations for the values of u_α at all interior points. The boundary points are only used to establish boundary conditions. For any function on this grid we use the notation $F(\mu_i, \nu_j, \tau^n) = F^n_{i,j}$. Time derivatives are approximated by forward difference quotients and velocity derivatives by centered differences,

$$\begin{aligned}
\left(\frac{\partial F}{\partial \tau}\right)_{i,j} &\approx \delta_\tau F = \frac{1}{\Delta\tau}(F^{n+1}_{i,j} - F^n_{i,j}), \\
\left(\frac{\partial F}{\partial \nu}\right)^n_{i,j} &\approx \delta_\nu F = \frac{1}{2\,\Delta\nu}(F^n_{i,j+1} - F^n_{i,j-1}), \\
\left(\frac{\partial^2 F}{\partial \nu^2}\right)^n_{i,j} &\approx \delta_\nu{}^2 F = \frac{1}{(\Delta\nu)^2}(F^n_{i,j+1} - 2F^n_{i,j} + F^n_{i,j-1}), \\
\left(\frac{\partial F}{\partial \mu}\right)^n_{i,j} &\approx \delta_\mu F = \frac{1}{2\Delta\mu}(F^n_{i+1,j} - F^n_{i-1,j}), \\
\left(\frac{\partial^2 F}{\partial \mu^2}\right)^n_{i,j} &\approx \delta_\mu{}^2 F = \frac{1}{(\Delta\mu)^2}(F^n_{i+1,j} - 2F^n_{i,j} + F^n_{i-1,j}), \\
\left(\frac{\partial^2 F}{\partial \nu\,\partial \mu}\right)^n_{i,j} &\approx \delta_\nu\delta_\mu F = \frac{1}{4\Delta\nu\,\Delta\mu}(F^n_{i+1,j+1} - F^n_{i-1,j+1} - F^n_{i+1,j-1} + F^n_{i-1,j-1}).
\end{aligned} \tag{19}$$

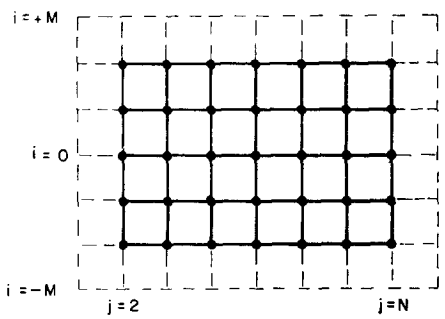

FIG. 1. Finite difference grid for Eq. (21).

Integrals are approximated by finite sums over the set of solution points,

$$\int_{-1}^{+1}\int_{0}^{\infty} F(\mu, \nu)\, d\mu\, d\nu \approx \sum_{i=-M+1}^{M-1}\sum_{j=2}^{N} F_{i,j}\, \Delta\nu\, \Delta\mu,$$

$$\int_{0}^{\nu} F(\mu, \nu)\, d\nu \approx \sum_{k=2}^{j} F_{i,k}\, \Delta\nu, \tag{20}$$

$$\int_{\nu}^{\infty} F(\mu, \nu)\, d\nu \approx \sum_{k=j}^{N} F_{i,k}\, \Delta\nu.$$

Using these definitions we can write Eq. (10) in the finite difference form

$$\delta_\tau(u_\alpha) = \delta_\nu{}^2(u_\alpha K_\alpha)^{n+1} + \delta_\nu\left(\frac{u_\alpha K_\alpha}{\nu}\right)^{n+1} - \delta_\nu\left(\frac{u_\alpha P_\alpha}{\nu}\right)^{n+1} + \delta_\mu(u_\alpha D_\alpha)^{n+1}$$
$$+ \delta_\mu{}^2(u_\alpha R_\alpha)^{n+1} + \delta_\nu\delta_\mu(u_\alpha S_\alpha)^{n+1}. \tag{21}$$

The coefficients in Eq. (21) are obtained from (15) and (16) by truncating the polynomial expansions to a finite number of terms and computing integrals according to (20). For example, Eq. (15a) becomes

$$K_{\alpha_{i,j}}^{n+1} = \sum_{l=0}^{L} P_l(\mu_i) \sum_{\beta} r_{\alpha\beta}\, \frac{1}{2}\left(\frac{\partial^2}{\partial\nu^2}\, g_\beta{}^l\right)_j^{n+1},$$

where, from (16c),

$$\left(\frac{\partial^2}{\partial\nu^2}\, g_\beta{}^l\right)_j^{n+1} = \frac{4\pi}{(2l+1)}\left\{\frac{1}{(2l+3)}\left[(l+2)\,\nu_j^{l+1}\sum_{k=j}^{N}\nu_k^{-l-1}(u_\beta{}^l)_k^{n+1}\,\Delta\nu\right.\right.$$
$$\left. - (l+1)\,\nu_j^{-l-2}\sum_{k=2}^{j}\nu_k^{l+2}(u_\beta{}^l)_k^{n+1}\,\Delta\nu\right]$$
$$\left. - \frac{1}{(2l-1)}\left[l\nu_j^{l-1}\sum_{k=j}^{N}\nu_k^{1-l}(u_\beta{}^l)_k^{n+1}\,\Delta\nu - (l-1)\,\nu_j^{-l}\sum_{k=2}^{j}\nu_k{}^l(u_\beta{}^l)_k^{n+1}\,\Delta\nu\right]\right\}.$$

The quantities $(u_\beta{}^l)_k^{n+1}$ are obtained from Eq. (14),

$$(u_\beta{}^l)_k^{n+1} = \frac{2l+1}{2}\sum_{r=-M+1}^{M-1} P_l(\mu_r)(u_\beta)_{k,r}^{n+1}\Delta\mu. \tag{22}$$

The other coefficients of Eq. (21) are computed in the same way.

The integer L used to truncate the coefficient expansions is chosen large enough

that terms of higher order have negligible value. In practice this number will depend on the anisotropy of the distribution functions. We shall prove in the next section that the mass and energy conservation properties of the scheme (21) are completely independent of the number of terms taken in the expansions. In order that the scheme conserve momentum, however, one must choose a value of L large enough that higher-order terms are negligible.

It is important to note that in the scheme (21) both the distribution functions and the Fokker–Planck coefficients are computed on the same time level. As we shall see in the next section, this is necessary for conservation of momentum and energy (but not for mass conservation). Equation (21) would also be conservative if written in purely explicit form, with all quantities on the right evaluated at time step n. The implicit form is preferred for most problems because it is stable regardless of the time step.

Proof of Conservation Properties

We now prove that the scheme (21) conserves the integrals for total system mass, energy, and momentum throughout the interior of velocity space independent of $\Delta\nu$ and $\Delta\mu$. The only errors will then be from points on the boundary. Although the proof can be given for a plasma with any number of charged species the algebra required tends to obscure the results. Thus, for clarity, we shall give the proof assuming that all particles have identical mass. Hereafter we set $\alpha = \beta$ and omit species subscripts. The time superscripts will also be deleted. Summations over the interior points (18) will be abbreviated as

$$\sum_{i=-M+1}^{M-1} \sum_{j=2}^{N} = \sum_{i} \sum_{j}.$$

Mass conservation

To show that the system (21) conserves total mass, we integrate the collision operator over the velocity space (18),

$$\sum_{i} \sum_{j} C_{i,j}\, \Delta\nu\, \Delta\mu = \Delta\nu\, \Delta\mu \sum_{i} \sum_{j} \left\{ \delta_\nu^2(uK) + \delta_\nu \left(\frac{uK}{\nu}\right) - \delta_\nu \left(\frac{uP}{\nu}\right) + \delta_\mu(uD) \right.$$

$$\left. + \delta^2(uR) + \delta_\mu \delta_\nu(uS) \right\}. \tag{23}$$

Because the difference operators in Eq. (23) are all centered, each term on the

right becomes a simple telescoping series which vanishes identically for all points interior to the region (18); that is, for

$$i = -M + 2,..., M - 2,$$

$$j = 3,..., N.$$

The summation (23) is therefore zero except for a contribution from the boundary points of (17) and their adjacent interior points. Thus we have

$$\Delta\nu\, \Delta\mu \sum_i \sum_j C_{i,j} = R_b\,, \tag{24}$$

where R_b is called a boundary remainder.

Energy Conservation

To show that (21) conserves total energy inside the region (18), we multiply by ν_j^2 and integrate,

$$\sum_i \sum_j \nu_j^2 C_{i,j}\, \Delta\nu\, \Delta\mu = \sum_i \sum_j \nu_j^2 \left\{\delta_\nu^2(uK) + \delta_\nu\left(\frac{uK}{\nu}\right) - \delta_\nu\left(\frac{uP}{\nu}\right) + \delta_\mu^2(uR) + \delta_\mu(uD) + \delta_\mu\delta_\nu(uS)\right\} \Delta\nu\, \Delta\mu.$$

Using the definitions (19) and the relation $\nu_{j+1}^2 - \nu_{j-1}^2 = 4\nu_j\Delta\nu$, we can reduce this to

$$\sum_i \sum_j \nu_j^2 C_{i,j}\, \Delta\nu\, \Delta\mu = \sum_i \sum_j 2(uP)_{i,j}\, \Delta\nu\, \Delta\mu + Q_b\,, \tag{25}$$

where Q_b is a remainder which only involves boundary points. The term containing $(uP)_{i,j}$ is evaluated from Eqs. (15), (16), and (20),

$$\sum_i \sum_j 2(uP)_{i,j}\, \Delta\nu\, \Delta\mu = 8\pi \sum_{l=0}^{L} \sum_{i=-M+1}^{M-1} \left[\sum_{j=2}^{N} \sum_{k=j}^{N} \nu_j{}^l \nu_k^{-l-1} u_k{}^l u_{i,j} P_l(\mu_i) - \sum_{j=2}^{N} \sum_{k=2}^{j} \nu_k{}^l \nu_j^{-l-1} u_k{}^l u_{i,j} P_l(\mu_i)\right] \Delta\mu(\Delta\nu)^2.$$

Using the definition (22) and exchanging the dummy j and k indices in the second double summation, we find

$$\sum_i \sum_j 2(uP)_{i,j}\, \Delta\mu\, \Delta\nu = 8\pi \sum_{l=0}^{L} \frac{2}{2l+1} \left[\sum_{j=2}^{N} \sum_{k=j}^{N} - \sum_{k=2}^{N} \sum_{j=2}^{k}\right] \nu_j{}^l \nu_k^{-l-1} u_j{}^l u_k{}^l (\Delta\nu)^2.$$

This expression is identically zero for every value of l because the two double summations cover the same set of points. Therefore energy is conserved except for a remainder Q_b due to boundary points.

Momentum Conservation

Because we are considering azimuthally symmetric distribution functions, the only component of macroscopic momentum is along the polar axis. Multiplying Eq. (21) by $\mu_i \nu_j \Delta\nu\, \Delta\mu$ and using the definitions (19), we obtain

$$\sum_i \sum_j \mu_i \nu_j C_{i,j}\, \Delta\mu\, \Delta\nu = \sum_i \sum_j u_{i,j} \left[-\frac{\mu_i}{\nu_j} K_{i,j} + \frac{\mu_i}{\nu_j} P_{i,j} - \nu_j D_{i,j} + S_{i,j} \right] \Delta\mu\, \Delta\nu + M_b , \tag{26}$$

where M_b is a remainder involving boundary points. To evaluate (26) we first use Eqs. (11) to obtain

$$\sum_i \sum_j \mu_i \nu_j C_{i,j}\, \Delta\mu\, \Delta\nu = \Delta\mu\, \Delta\nu \sum_i \sum_j T_{i,j} + M_b ,$$

where

$$T_{i,j} = u_{i,j} \left[\mu_i \left(\frac{\partial H}{\partial \nu} \right)_{i,j} + \frac{(1 - \mu_i^2)}{\nu_j} \left(\frac{\partial H}{\partial \mu} \right)_{i,j} \right].$$

The quantity $T_{i,j}$ is obtained from Eqs. (15), (16), and the identities [14]

$$(1 - \mu^2) \frac{\partial}{\partial \mu} P_l = \frac{l(l+1)}{(2l+1)} [P_{l-1} - P_{l+1}],$$

$$(2l + 1)\, \mu P_l(\mu) = (l + 1)\, P_{l+1} + l P_{l-1} ;$$

we then find

$$\Delta\nu\, \Delta\mu \sum_i \sum_j T_{i,j} = \Delta\nu \sum_{j=2}^{N} \sum_{l=0}^{L} \frac{4\pi}{2l+1} \left\{ \left(\sum_{-M+1}^{M-1} P_{l-1}(\mu_i)\, u_{i,j}\, \Delta\mu \right) l \nu_j^{l-1} \sum_{k=j}^{N} \nu_k^{-l-1} u_k{}^l\, \Delta\nu \right.$$

$$\left. - \left(\sum_{-M+1}^{M-1} P_{l+1}(\mu_i)\, u_{i,j}\, \Delta\mu \right) (l+1)\, \nu_j^{-l-2} \sum_{k=2}^{j} \nu_k{}^l u_k{}^l\, \Delta\nu \right\}.$$

Now if we renumber the terms in the second double summation so that $l \to l - 1$,

exchange dummy j and k indices in this term, and apply the definition (22), we find

$$\Delta v\,\Delta\mu \sum_i \sum_j T_{i,j}$$

$$= 4\pi \left\{ \sum_{l=1}^{L} \frac{2l}{(2l+1)(2l-1)} \left(\sum_{j=2}^{N} \sum_{k=j}^{N} - \sum_{k=2}^{N} \sum_{j=2}^{k} \right) v_j^{l-1} v_k^{-l-1} u_j^{l-1} u_k^{l} (\Delta v)^2 \right\}$$

$$- 4\pi \left\{ \frac{2(L+1)}{(2L+1)(2L+3)} \sum_{k=2}^{N} \sum_{j=2}^{k} v_j^{L} v_k^{-L-2} u_j^{L} u_k^{L+1} (\Delta v)^2 \right\}.$$

The first quantity in brackets is identically zero for every value of l because the two double summations on j and k cover the same set of points. The second quantity in brackets is negligably small if L is chosen large enough. Consequently, (21) conserves momentum inside the region (18).

3. Reduction to a Set of One-Dimensional Problems

The difference scheme of the previous section is particularly suitable for treating long-time problems in a highly anisotropic plasma. However, it is a two-dimensional system and requires a large amount of computation. In many cases the plasma is not highly anisotropic and the integration covers only a few collision times. For this class of problems it is possible to achieve a great simplification by expanding the angular dependence of the collision operator in Legendre polynomials. This expansion reduces Eq. (10) to an infinite system of one-dimensional problems. When the distribution functions are close to spherical symmetry the system can be truncated to a small number of terms, thus greatly reducing the amount of computation.

Expansion of the collision operator was suggested in a paper by Rosenbluth, MacDonald, and Judd [12], but the reduced equations were not derived. The zeroth- and first-order equations have been given by several authors [8–10]. Attempts to find the general term have been made by Shkarofsky [15] and Oliphant [16]. In order to obtain manageable expressions, Shkarofsky neglects products of electron and ion distribution functions above first order. Oliphant's results are only formal because the final equations are given in terms of complicated angular integrals which cannot be evaluated in closed form.

By starting with our Eq. (10) we can obtain the exact expansion of the collision operator to all orders. No complicated integrations are required and the resulting system of one-dimensional equations has a very compact form. The more general three-dimensional case is a direct extension of this work.

Expansion of the Collision Operator

Suppose that the collision operator C_α and the distribution functions u_α are both expanded in Legendre polynomials,

$$u_\alpha(\nu, \mu, \tau) = \sum_{l=0}^{\infty} P_l(\mu)\, u_\alpha{}^l(\nu, \tau), \tag{27a}$$

$$C_\alpha(\nu, \mu, \tau) = \sum_{j=0}^{\infty} P_j(\mu)\, C_\alpha{}^j(\nu, \tau). \tag{27b}$$

Then by the orthogonality of the Legendre polynomials the Fokker–Planck equation becomes

$$\frac{\partial}{\partial \tau} u_\alpha{}^l(\nu, \tau) = C_\alpha{}^l(\nu, \tau), \qquad l = 0, 1, 2, \ldots . \tag{28}$$

Our object is to find the components C^l. We procede by first expanding the Fokker–Planck coefficients (11) completely in terms of Legendre polynomials. The terms $u_\alpha K_\alpha$, etc., which appear in Eq. (10), then become products of infinite series. The product series are reduced to simple series with Clebsch–Gordon coefficients, and the general term of (27b) is sorted out using summation diagrams.

The expansions of the Fokker–Planck coefficients are obtained from Eqs. (15) by renumbering terms in the series and using the recursion relation [14]

$$(2l + 1)\, \mu P_l(\mu) = (l + 1)\, P_{l+1} - l P_{l-1}\,.$$

We find

$$\begin{aligned}
K_\alpha(\nu, \mu) &= \sum_{l=0}^{\infty} P_l(\mu)\, K_\alpha{}^l(\nu), \\
P_\alpha(\nu, \mu) &= \sum_{l=0}^{\infty} P_l(\mu)\, \Pi_\alpha{}^l(\nu), \\
D_\alpha(\nu, \mu) &= \sum_{l=0}^{\infty} P_l(\mu)\, D_\alpha{}^l(\nu), \\
R_\alpha(\nu, \mu) &= \sum_{l=0}^{\infty} P_l(\mu)\, R_\alpha{}^l(\nu), \\
S_\alpha(\nu, \mu) &= \sum_{l=0}^{\infty} P_l(\mu)\, S_\alpha{}^l(\nu),
\end{aligned} \tag{29}$$

where

$$
\begin{aligned}
K_\alpha{}^l(\nu) &= \sum_\beta \frac{1}{2}\, r_{\alpha\beta}\, \frac{\partial^2}{\partial \nu^2}\, g_\beta{}^l, \\
\Pi_\alpha{}^l(\nu) &= \sum_\beta \left[k_{\alpha\beta\nu}\, \frac{\partial}{\partial \nu}\, h_\beta{}^l + r_{\alpha\beta} h_\beta{}^l \right], \\
D_\alpha{}^l(\nu) &= \Phi_\alpha^{l+1} - \Phi_\alpha^{l-1} + l\Psi_\alpha^{l-1} + (l+1)\, \Psi_\alpha^{l+1}, \\
\Phi_\alpha{}^l(\nu) &= \frac{l(l+1)}{(2l+1)} \sum_\beta \left[r_{\alpha\beta} \left(\frac{1}{\nu^3}\, \frac{\partial}{\partial \nu}\, g_\beta{}^l - \frac{1}{\nu^4}\, g_\beta{}^l \right) - k_{\alpha\beta}\, \frac{1}{\nu^2}\, h_\beta{}^l \right], \\
\Psi_\alpha{}^l(\nu) &= \frac{1}{(2l+1)} \sum_\beta r_{\alpha\beta} \left[\frac{1}{\nu^2}\, h_\beta{}^l - \frac{1}{2\nu^2}\, \frac{\partial^2}{\partial \nu^2}\, g_\beta{}^l \right], \\
R_\alpha{}^l(\nu) &= \Gamma_\alpha^{l+2} + \Sigma_\alpha{}^l + \Lambda_\alpha^{l-2}, \\
\Gamma_\alpha{}^l(\nu) &= \frac{l(l-1)}{(2l+1)(2l-1)} \sum_\beta r_{\alpha\beta} \left[\frac{(l+1)^2}{2\nu^4}\, g_\beta{}^l - \frac{1}{2\nu^3}\, \frac{\partial}{\partial \nu}\, g_\beta{}^l \right], \\
\Lambda_\alpha{}^l(\nu) &= \frac{(l+1)(l+2)}{(2l+1)(2l+3)} \sum_\beta r_{\alpha\beta} \left[\frac{l^2}{2\nu^4}\, g_\beta{}^l - \frac{1}{2\nu^3}\, \frac{\partial}{\partial \nu}\, g_\beta{}^l \right], \\
\Sigma_\alpha{}^l(\nu) &= \sum_\beta r_{\alpha\beta} \left[\left(1 - \frac{(l+1)^2}{(2l+1)(2l+3)} - \frac{l^2}{(2l+1)(2l-1)} \right) \frac{1}{2\nu^3}\, \frac{\partial}{\partial \nu}\, g_\beta{}^l \right. \\
&\quad \left. + \left(\frac{l(l+1)}{(2l-1)(2l+3)} - \frac{l^2(l^2-1)}{(2l+1)(2l-1)} - \frac{l(l+1)^2\,(l+2)}{(2l+1)(2l+3)} \right) \frac{1}{2\nu^4}\, g_\beta{}^l \right] \\
S_\alpha{}^l(\nu) &= \psi_\alpha^{l+1} - \psi_\alpha^{l-1}, \\
\psi_\alpha{}^l(\nu) &= \frac{l(l+1)}{(2l+1)} \sum_\beta r_{\alpha\beta} \left[\frac{1}{\nu^2}\, \frac{\partial}{\partial \nu}\, g_\beta{}^l - \frac{1}{\nu^2}\, g_\beta{}^l \right].
\end{aligned}
\tag{30}
$$

From Eqs. (27a) and (29), the terms of Eq. (10) become

$$
\begin{aligned}
u_\alpha K_\alpha &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P_l(\mu)\, P_k(\mu)\, u_\alpha{}^k(\nu)\, K_\alpha{}^l(\nu), \\
u_\alpha P_\alpha &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} P_l(\mu)\, P_k(\mu)\, u_\alpha{}^k(\nu)\, \Pi_\alpha{}^l(\nu), \\
&\vdots \\
&\text{etc.}
\end{aligned}
\tag{31}
$$

These product series can be reduced to simple series by the relation

$$P_k P_l = \sum_{j=|k-l|}^{|k+l|} c_{jkl} P_j \,. \tag{32}$$

The constants c_{jkl} are the squares of the Clebsch–Gordon coefficients [17],

$$c_{jkl} = (2j+1)\frac{g!\,(j+k-l)!\,(k+l-j)!\,(l+j-k)!}{(2g+1)!\,(g-j)!^2\,(g-k)!^2\,(g-l)!^2}, \tag{33}$$

where

$$g = \tfrac{1}{2}(j+k+l).$$

The c_{jkl} are symmetric in the first two indices and are zero unless g is integral and $|k+l| \geqslant j \geqslant |k-l|$. Equations (31) now become

$$u_\alpha K_\alpha = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{j=|k-l|}^{|k+l|} c_{jkl} P_j u_\alpha{}^k K_\alpha{}^l, \tag{34}$$

$$\vdots$$

etc.

To sort out the P_j terms from (34) we use the summation diagram, Fig. 2. This diagram is a map of the (j, l) plane showing all points covered for a fixed value of k. From the figure we see that

$$\sum_{l=0}^{\infty}\sum_{j=|k-l|}^{|k+l|} = \sum_{j=0}^{\infty}\sum_{l=|k-j|}^{|k+j|}.$$

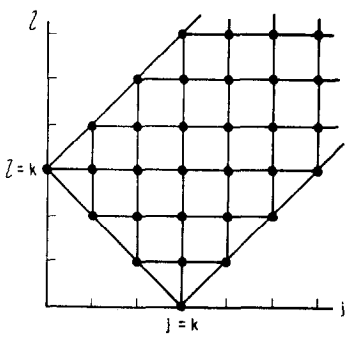

FIG. 2. Summation diagram for Eq. (39).

Thus Eq. (34) can be written

$$u_\alpha K_\alpha = \sum_{j=0}^{\infty} P_j \Big[\sum_{k=0}^{\infty} \sum_{l=|k-j|}^{|k+j|} c_{jkl} u_\alpha{}^k K_\alpha{}^l \Big].$$

Since all the expressions (31) have this form, we define the notation

$$\begin{aligned} [uK]_\alpha^j &= \sum_{k=0}^{\infty} \sum_{l=|k-j|}^{|k+j|} c_{jkl} u_\alpha{}^k K_\alpha{}^l, \\ [uP]_\alpha^j &= \sum_{k=0}^{\infty} \sum_{l=|k-j|}^{|k+j|} c_{jkl} u_\alpha{}^k \Pi_\alpha{}^l, \\ &\vdots \\ &\text{etc.} \end{aligned} \tag{35}$$

Then the Fokker–Planck equation (10) can be written

$$\sum_{k=0}^{\infty} P_k \frac{\partial}{\partial \tau} u_\alpha{}^k(\nu, \tau) = \sum_{j=0}^{\infty} P_j U_\alpha{}^j + \sum_{j=1}^{\infty} \Big(\frac{\partial}{\partial \mu} P_j \Big) V_\alpha{}^j + \sum_{j=2}^{\infty} \Big(\frac{\partial^2}{\partial \mu^2} P_j \Big) W_\alpha{}^j, \tag{36}$$

where

$$\begin{aligned} U_\alpha{}^j(\nu) &= \frac{\partial^2}{\partial \nu^2} [uK]_\alpha^j + \frac{\partial}{\partial \nu} \Big(\frac{[uK]_\alpha^j}{\nu} \Big) - \frac{\partial}{\partial \nu} \Big(\frac{[uP]_\alpha^j}{\nu} \Big), \\ V_\alpha{}^j(\nu) &= [uD]_\alpha^j + \frac{\partial}{\partial \nu} [uS]_\alpha^j, \\ W_\alpha{}^j(\nu) &= [uR]_\alpha^j . \end{aligned} \tag{37}$$

To complete the reduction of the Fokker–Planck equation we must expand the derivatives $\partial P_j/\partial \mu$ and $\partial^2 P_j/\partial \mu^2$ in Legendre polynomials and sort out the P_j terms from (36). First consider the term

$$\sum_{j=0}^{\infty} \Big(\frac{\partial}{\partial \mu} P_j \Big) V_\alpha{}^j \tag{38}$$

on the right side of Eq. (36). From the recursion relation [14]

$$\frac{\partial}{\partial \mu} P_{l+1} - \frac{\partial}{\partial \mu} P_{l-1} = (2l + 1) P_l$$

it follows that

$$\frac{\partial}{\partial \mu} P_j = \sum_{k=0,2,4}^{j-1} (1 + 2k)\, P_k \qquad \text{if } j \text{ is } \textit{odd},$$

$$\frac{\partial}{\partial \mu} P_j = \sum_{k=1,3,5}^{j-1} (1 + 2k)\, P_k \qquad \text{if } j \text{ is } \textit{even}.$$

Thus the expression (38) is

$$\left(\sum_{j=1,3,5}^{\infty} \sum_{k=0,2,4}^{j-1} + \sum_{j=2,4,6}^{\infty} \sum_{k=1,3,5}^{j-1} \right) (1 + 2k)\, P_k V_\alpha{}^j.$$

Now using the summation diagram, Fig. 3, we can rewrite this as

$$\left(\sum_{k=0,2,4}^{\infty} \sum_{j=k+1,k+3}^{\infty} + \sum_{k=1,3,5}^{\infty} \sum_{j=k+1,k+3}^{\infty} \right) (1 + 2k)\, P_k V_\alpha{}^j.$$

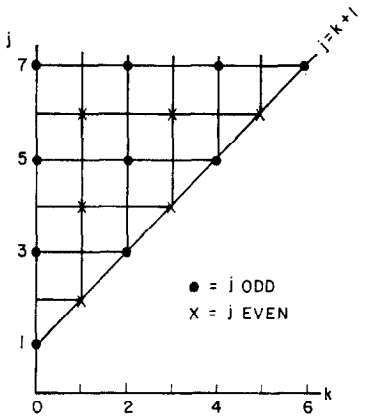

FIG. 3. Summation diagram for Eq. (38).

Exchanging the dummy j and k indices and adding, we find

$$\sum_{j=1,2,3}^{\infty} \left(\frac{\partial}{\partial \mu} P_j \right) V_\alpha{}^j = \sum_{j=0,1,2}^{\infty} (1 + 2j)\, P_j \sum_{k=j+1,j+3}^{\infty} V_\alpha{}^k. \tag{39}$$

The second derivative term of Eq. (36) can be handled in a similar way. Using Eq. (39) we first write

$$\sum_{j=1}^{\infty} \left(\frac{\partial}{\partial \mu} P_j\right) W_\alpha^j = \sum_{j=0}^{\infty} P_j \Phi_\alpha^j, \tag{40}$$

where

$$\Phi_\alpha^j(\nu) = (1 + 2j) \sum_{k=j+1,j+3}^{\infty} W_\alpha^k;$$

differentiating Eq. (40) and using (39) a second time, we have

$$\sum_{j=2}^{\infty} \left(\frac{\partial^2}{\partial \mu^2} P_j\right) W_\alpha^j = \sum_{j=0}^{\infty} P_j(1 + 2j) \sum_{k=j+1,j+3}^{\infty} \Phi_\alpha^k. \tag{41}$$

To simplify Eq. (41), consider the double summation

$$\sum_{k=j+1,j+3}^{\infty} \Phi_\alpha^k = \sum_{k=j+1,j+3}^{\infty} \sum_{l=k+1,k+3}^{\infty} (1 + 2k)\, W_\alpha^l. \tag{42}$$

Using the summation diagram, Fig. 4, the right side of Eq. (42) becomes

$$\sum_{l=j+2,j+4}^{\infty} W_\alpha^l \left(\sum_{k=j+1,j+3}^{l-1} (1 + 2k) \right) = \sum_{l=j+2,j+4}^{\infty} \tfrac{1}{2}(l - j)(l + j + 1)\, W_\alpha^l. \tag{43}$$

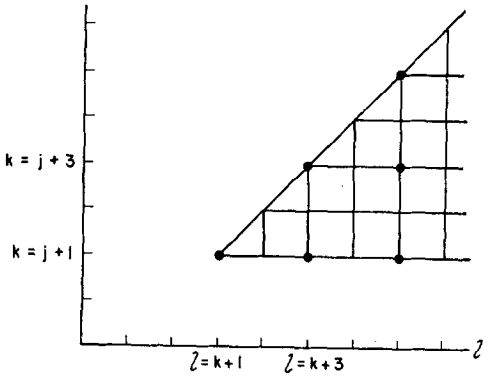

FIG. 4. Summation diagram for Eq. (42).

Now substituting Eqs. (39) and (43) into (36) we have

$$\frac{\partial}{\partial \tau} u_\alpha{}^j(\nu, \tau) = U_\alpha{}^j + (1 + 2j) \sum_{k=j+1, j+3}^{\infty} V_\alpha{}^k + (1 + 2j) \sum_{k=j+2, j+4}^{\infty} W_\alpha{}^k \tfrac{1}{2}(k - j)(k + j + 1). \tag{44}$$

Explicit Equations

Equation (44) is the exact expansion of the Fokker–Planck equation (10). To treat this system numerically we keep components u^l up to some order L and assume that all higher terms are negligable. The system is then written out explicitly by using equations (30), (33), and (35). The first three equations of the chain are (omitting species subscripts)

$$\begin{aligned}\frac{\partial}{\partial \tau} u^0 = {} & \frac{\partial^2}{\partial \nu^2} \left\{u^0 K^0 + \frac{1}{3} u^1 K^1 + \frac{1}{5} u^2 K^2\right\} \\ & + \frac{\partial}{\partial \nu} \left\{\frac{1}{\nu} \left[u^0(K^0 - \Pi^0) + \frac{1}{3} u^1(K^1 - \Pi^1) + \frac{1}{5} u^2(K^2 - \Pi^2)\right]\right\} \\ & + \frac{\partial}{\partial \nu} \{u^0(S^1 + S^3) + u^1(S^0 + S^2) + u^2(S^1 + 0.27222 S^3)\} \\ & + u^0[D^1 + D^3 + 3R^2 + 3R^4] + u^1[D^0 + D^2 + 2R^1 + 1.5238 R^3] \\ & + u^2 \left[D^1 + 3R^0 + \frac{6}{7} R^2 + 0.27222 D^3 + 0.1777 R^4\right],\end{aligned} \tag{45a}$$

$$\begin{aligned}\frac{\partial}{\partial \tau} u^1 = {} & \frac{\partial^2}{\partial \nu^2} \left\{u^0 K^1 + u^1 \left(K^0 + \frac{2}{5} K^2\right) + \frac{2}{5} u^2 K^1\right\} \\ & + \frac{\partial}{\partial \nu} \left\{\frac{1}{\nu} \left[u^0(K^1 - \Pi^1) + u^1 \left(K^0 - \Pi^0 + \frac{2}{5} K^2 - \frac{2}{5} \Pi^2\right) + \frac{2}{5} u^2(K^1 - \Pi^1)\right]\right\} \\ & + \frac{\partial}{\partial \nu} \left\{3u^0 S^2 + u^1(2S^2 + 1.3571 S^3) + u^2 \left(3S^0 + \frac{6}{7} S^2\right)\right\} \\ & + u^0(3D^2 + 15R^3) + u^1(2D^1 + 9R^2 + 1.3571 D^3 + 0.31666 R^4) \\ & + u^2 \left(3D^0 + 9R^1 + \frac{6}{7} D^2 + \frac{1}{3} R^3\right),\end{aligned} \tag{45b}$$

$$\frac{\partial}{\partial \tau} u^2 = \frac{\partial^2}{\partial v^2}\left\{u^0 K^2 + \frac{2}{3} u^1 K^1 + u^2\left(K^0 + \frac{2}{7} K^2\right)\right\}$$

$$+ \frac{\partial}{\partial v}\left\{\frac{1}{v}\left[u^0(K^2 - \Pi^2) + \frac{2}{3} u^1(K^1 - \Pi^1) + u^2\left(K^0 - \Pi^0 + \frac{2}{7} K^2 - \frac{2}{7}\Pi^2\right)\right]\right.$$

$$+ \frac{\partial}{\partial v}\{5u^0 S^3 + 3u^1 S^2 + u^2(3S^1 + 0.07539S^3)\}$$

$$+ u^0(5D^3 + 35R^4) + u^1\left(3D^2 + \frac{5}{6} R^3\right)$$

$$+ u^2(3D^1 + 3R^2 + 0.07539D^3 + 0.1325R^4), \tag{45c}$$

where u^3, u^4,... are assumed negligibly small. In computing the coefficients K^l, Π^l, D^l, etc. from Eq. (30) we omit components higher than $l = 2$.

When the distribution functions are very close to spherical symmetry so that $u^0 \gg u^1 \gg u^2$, etc., the system (44) can be simplified by dropping terms of higher order in each equation. The equations are then decoupled and all equations above zeroth order are linear.

The one-dimensional system (44) can be converted to finite difference form with the discrete approximations (19) and (20). The resulting difference equations will not rigorously satisfy the conservation laws except when the distribution functions have spherical symmetry. In that case the zeroth-order equation (45a) and the collision operator (10) are identical and both methods will conserve mass and energy.

4. Relaxation of a Deuterium–Electron Plasma

As an application of our numerical methods we compute the thermal equipartition of a deuterium plasma in which the electron and ion temperatures are initially unequal. This problem was first considered by Killeen, Heckrotte, and Boer [4]. These authors assumed the deuteron distribution to be stationary in time and only solved the electron Fokker–Planck equation. Their results are therefore valid only for times much shorter than the equipartition time. They found that the energy transfer rate from hot deuterons to colder electrons was somewhat slower than predicted by the Maxwellian approximation of Spitzer [11]. In a subsequent paper, House and Swartztrauber [5] solved coupled Fokker–Planck equations for both ions and electrons. They had serious difficulty satisfying the conservation laws and were only able to obtain solutions for about one D–D collision time. It is therefore difficult to draw conclusions about the equipartition time from their paper.

Our object is to obtain long-time solutions of the Fokker–Planck equations for electrons and deuterons when $T_e \neq T_d$. For simplicity we consider isotropic

computed from the known values u_j^n. Then (47) was solved as a linear system to give provisional values for u_j^{n+1}. The provisional values so obtained were used to recompute the coefficients and, holding time constant, (47) was solved again. The cycle was repeated until coefficient values ceased to change. We then advanced to the next time step. In practice, three iterations per time step gave very accurate convergence.

The equations were solved for a large number of different initial electron and ion distribution functions with initial temperature ratios in the range $T_e/T_i = 0.01$ to $T_e/T_i = 100$. All these calculations lead to the same result: The rate of thermal equilibration between electrons and deuterons is virtually independent of the initial shape of the distribution functions and depends only on their energy.

To illustrate a typical calculation, we take the electron distribution to be Maxwellian and the deuteron distribution to be a nearly monoenergetic Gaussian with a temperature ten times that of the electrons,

$$u_e(v, 0) = v^2 \exp[-1.5v^2],$$
$$u_i(v, 0) = v^2 \exp[-(v - .052)^2/(.002)^2].$$

These functions were defined at 150 points on the velocity axis $v = 0$ to $v = 5$, where $v = 1$ corresponds to the thermal speed of electrons at $t = 0$. Because the deuteron distribution is much narrower than that of the electrons, the grid points were spaced closely together near $v = 0$ and much further apart near the tail of the electron distribution. The difference equations were solved through 1180 time steps using an increment $\Delta\tau = 15$ for the first 120 steps and $\Delta\tau = 25$ for the remainder. Problem time was 48 seconds on an IBM 360/95 computer. Total system mass was constant to within four parts in 10^{12}. Total energy of the system varied by less than five parts in 10^5. The error in mass conservation is due only to finite computer word size (the IBM/360 carries about 16 decimal digits). The energy conservation error is somewhat larger and reflects the cumulative size of the energy boundary remainder. Both these errors are negligible in the present problem.

Figure 5 shows the time history of the distribution functions. We have plotted the functions $f = u/v^2$ so that the final Maxwellian shape can be seen. The vertical scale is in arbitrary units. Figure 6 shows the temperature history of the plasma, computed from Eq. (49c). We can express the dimensionless time in seconds using Eq. (49a),

$$t = \left(0.60 \times 10^{23} \frac{[kT_e(0)]^{3/2}}{n_e \ln \Lambda}\right) \tau \text{ sec.} \tag{50}$$

For comparison, the e–e collision time given by Spitzer [11] is

$$t_{e-e} = 1.64 \times 10^{23} \frac{[kT_e]^{3/2}}{n_e \ln \Lambda} \text{ sec.}$$

Thus, initially, two units of τ correspond to one e–e collision time. However, t_{e-e} increases nearly tenfold as the electron energy increases. Tables of ln Λ given by Spitzer [11] allow the conversion of Eq. (50) to real time. Taking typical theta pinch parameters [5], $kT_e(0) = 300$ eV, $kT_i(0) = 3$ keV, and $n_e = 10^{16}$, each unit of τ equals 5.2 nsec and the total problem time, $\tau = 28{,}300$, is 1.48×10^{-4} sec. We note that the deuteron distribution has relaxed to Maxwellian by $\tau = 5300$ or 2.8×10^{-5} sec. This agrees well with the Spitzer D–D collision time [11], which

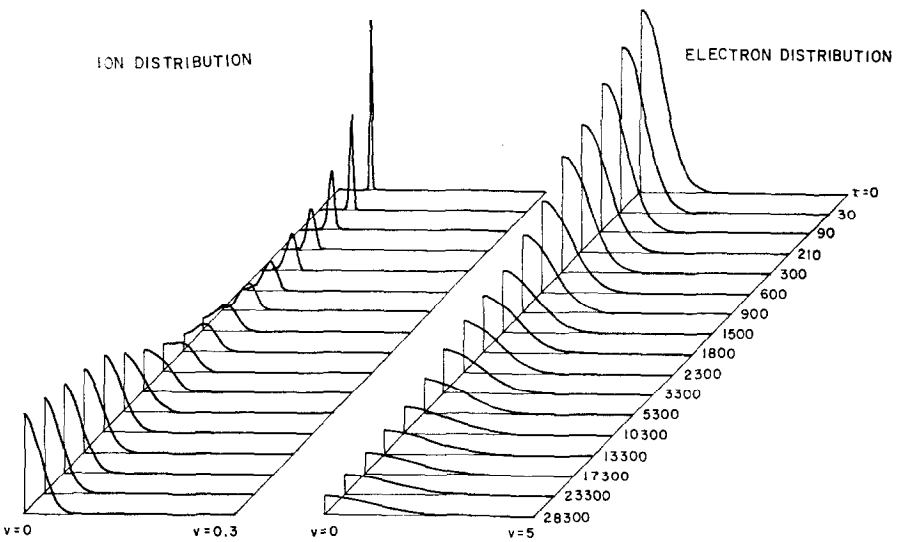

FIG. 5. Electron and ion distribution functions at different instants during the thermal equipartition. Vertical scale in arbitrary units.

is about 3×10^{-5} sec. House and Swartztrauber [5] for the same initial conditions give a deuteron relaxation time of 6μ sec, which is incorrect by nearly an order of magnitude.

It is interesting to compare the thermal relaxation rates obtained from the Fokker–Planck equation with a simple model proposed by Spitzer [11]. If one assumes that the distribution functions are always Maxwellian during the relaxation, then the temperatures T_e and T_i obey the equation

$$\frac{dT_e}{d\tau} = \frac{T_e - T_i}{[T_e + (m_e/m_i)\, T_i]^{3/2}}, \tag{51}$$

where the parameter τ is related to the time t by

$$\tau = \left(\frac{8}{3}\sqrt{2\pi}\,\frac{n_e e^4}{k^{3/2}}\,\frac{\sqrt{m_e}}{m_i}\right) t.$$

This equation can be integrated quite simply if we have $T_e + T_i = T_0 = \text{constant}$. In the present problem we can neglect the term $(m_e/m_i)\, T_i$ in the denominator of (51). Then the solution is

$$\tau = \frac{1}{3}\left[T_e^{3/2}(0) - T_e^{3/2}(\tau)\right] + \frac{1}{2}\, T_0\left[T_e^{1/2}(0) - T_e^{1/2}(\tau)\right] + \frac{1}{2}\left(\frac{T_0}{2}\right)^{3/2} \ln\left[\frac{\Phi(\tau)}{\Phi(0)}\right], \quad (52)$$

where

$$\Phi(\tau) = \{(T_0/2)^{1/2} + T_e^{1/2}(\tau)\}/\{(T_0/2)^{1/2} - T_e^{1/2}(\tau)\}.$$

The solution (52) is plotted alongside the Fokker–Planck results in Fig. 6. The greatest difference between the curves is 1.3 % of the electron temperature, occurring at $\tau = 210$. This remarkable agreement between the Spitzer model and the

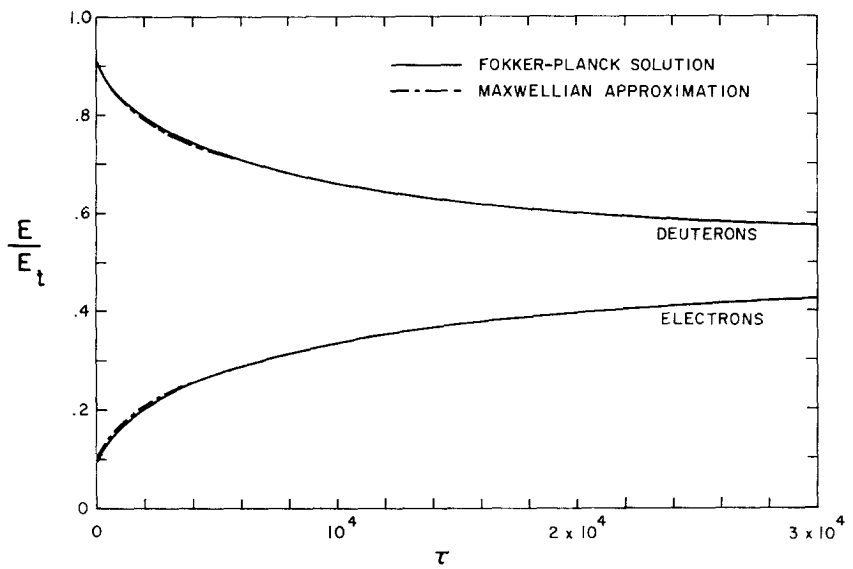

FIG. 6. Electron and ion thermal energy as a fraction of the total system energy E_t during the equipartition.

Fokker–Planck solutions held for all cases tested from $T_D/T_e = 100$ to $T_D/T_e = 0.01$. The maximum difference between the two models was 6 % of the electron temperature during a computation in which $T_D(0)/T_e(0) = 100$. The lower temperature was obtained from the Fokker–Planck equation. We conclude that the Spitzer model gives a simple and accurate means for calculating thermal relaxation in an isotropic plasma.

ACKNOWLEDGMENTS

The author wishes to thank Professor C. K. Chu for his constant helpful advice and criticism throughout this work. Thanks go to Professor W. T. Sanders who suggested the diagram method used in Section 3. I also wish to thank the Goddard Institute for Space Studies which made its excellent computer facilities available, and the National Science Foundation which supported this work under grant NSF GK-1258.

REFERENCES

1. W. M. MACDONALD, M. N. ROSENBLUTH, AND W. CHUCK, *Phys. Rev.* **107** (1957), 350.
2. H. DREICER, "Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy," Vol. 31, p. 57, Geneva, 1958.
3. G. BING AND J. ROBERTS, *Phys. Fluids* **4** (1961), 1039.
4. J. KILLEEN, W. HECKROTTE, AND G. BOER, *Nucl. Fusion, Suppl.*, Part 1 (1962), 183.
5. L. L. HOUSE AND P. N. SWARTZTRAUBER, *Phys. Fluids* **10** (1967), 605.
6. J. KILLEEN AND A. FUTCH, *J. Comput. Phys.* **2** (1968), 236.
7. K. D. MARX, Ph. D. Thesis, University of California at Davis, 1968.
8. R. S. COHEN, L. SPITZER, AND P. MCROUTLEY, *Phys. Rev.* **80** (1950), 230.
9. H. DREICER, *Phys. Rev.* **117** (1960), 343.
10. I. P. SHKAROFSKY, *Can. J. Phys.* **39** (1961), 1619.
11. L. SPITZER, "Physics of Fully Ionized Gases," 2nd ed., Chapter 5, Interscience Publishers, Inc., New York, 1962.
12. M. N. ROSENBLUTH, W. M. MACDONALD, AND D. L. JUDD, *Phys. Rev.* **107** (1957), 1.
13. J. C. WHITNEY, Ph. D. Thesis, Columbia University, 1969.
14. J. D. JACKSON, "Classical Electrodynamics," Chapter 3, John Wiley and Sons, New York
16. T. A. OLIPHANT, Los Alamos Scientific Laboratory Report LA-3437.
17. A. MESSIAH, "Quantum Mechanics," Vol. II, p. 1057, North Holland, Amsterdam, 1965.